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Abstract

This study is a short note designed to underline the importance of using the theoretically required form of accumulation functions. It is now a common knowledge that a growth model must rely on non-diminishing returns to a factor of production in order to generate endogenous growth. In Lucas (1988), for example, there is no diminishing-returns to the accumulation of human capital, which is the source of endogenous growth in the model. This rule, however, can lead to the following potentially misleading assumption: diminishing marginal productivity to each factor of production—given that there is no other source of long run growth—is sufficient for generating steady state equilibrium at levels. In this short note, we make two points. First, diminishing marginal productivity alone is not necessarily sufficient for generating steady state equilibrium at levels. Second, the inclusion of a theoretically required counter-force in the accumulation function together with diminishing returns is sufficient for generating steady state equilibrium. In conclusion, we heuristically argue that an accumulation function with no theoretically required counter-moving force, with or without diminishing returns, may bias the results of the model.

Key Words: Accumulation function, Stationary state, Steady state, Differential equations, Economic Growth, Long-run Equilibrium.

JEL Classification: O10, O15, O41.

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1 Introduction

In this short note, we underline the importance of using the true, that is, the theoretically required form of accumulation function in an economic growth model. Since Solow (1956), it is well-known that diminishing returns to each factor of production is necessary, but not in itself sufficient, for stationary equilibrium (at levels) in a dynamic model. In particular, an equation of motion must include 'counter-moving forces' in addition to diminishing returns in order to generate stationary equilibrium (see figures I-III and V in Solow (1956), emphasizing this point). We observe that the importance of this requirement is sometimes overlooked in the literature. We have a heuristic explanation for this tendency to overlook this factor: after the revival of economic growth theory in mid-1980s, the focus was on endogenous growth mechanisms, which, by and large, were based on non-diminishing returns to a factor of production. This fact may have led to the mis-conclusion that diminishing marginal productivity to each factor of production—given that there is no other source of long run growth—is sufficient for generating steady state equilibrium at levels. In this short paper, we use a particular example of Lucas (1988) model to show that the stationary equilibrium of a dynamic model necessarily requires a counter-moving force in equation(s) of motion, in addition to diminishing marginal return to individual factors of production. The organization of this note is as follows. Section 2 discusses several versions of the human capital accumulation function in Lucas (1988) and shows that (i) relying on diminishing marginal returns to each factor of production is not sufficient to ensure long run equilibrium, (ii) it is important not to ignore incorporating a theoretically required counter-moving force in the accumulation function, even when there are no diminishing-returns. Section 3 concludes the note.

2 A Demonstration

Let us first show the critical role played by a counter-moving force in a plain accumulation function, which cannot be secured by the very existence of diminishing returns. To this end, let us assume that we have the following accumulation function (you may consider this the very first part of Solovian fundamental equation of growth for a unitary saving rate):

$$\dot{h} = h^{\xi} \qquad 0 < \xi < 1 \tag{1}$$

The solution of this simple differential equation would yield

$$h(t) = \left[(1-\xi)\cdot t + h_0^{1-\xi}\right]^{\frac{1}{1-\xi}} \quad (2)$$

where h_0 is the initial value of stock of human capital. Notably, $Lim_{t\to\infty}h(t) = \infty$, that is, there is no steady state at levels to the differential equation. One may easily show that this differential equation has a zero growth rate at steady state, though it approaches infinity at levels. To see this, write (1) in growth form: $\frac{\dot{h}}{h} = h^{\xi-1}$. Substituting the solution of $h(t)$ from (2) into this growth equation, one can easily show that $Lim_{t\to\infty}\frac{\dot{h}}{h} = 0$. Hence, for single-equation accumulation functions, diminishing marginal returns ensures a stationary state, but not a long run equilibrium at level.

Let us now assume that we have a modified (1), in which there is a counter-moving force to h. For matter of illustration, if h is human capital per person, then $(n+\delta)$ is the effective depreciation of human capital (you may also consider this the complete Solovian fundamental equation of growth for a unitary saving rate):

$$\dot{h} = h^{\xi} - (n+\delta)h \quad (3)$$

where $0 < \delta < 1$ is the depreciation rate. The solution of the problem would yield

$$h(t) = \left[\frac{1}{n+\delta} + \left(h_0^{1-\xi} - \frac{1}{n+\delta}\right)e^{-(n+\delta)(1-\xi)t}\right]^{\frac{1}{1-\xi}} \quad (4)$$

which has a stationary state value at infinity, $Lim_{t\to\infty}h(t) = \left(\frac{1}{n+\delta}\right)^{\frac{1}{1-\xi}} \equiv h_{ss}$, where a subscript ss means steady-state. As one may note, the two results in (2) and (4) are *qualitatively different*. The latter has a steady-state at level and in growth rate; the former has only steady state growth rate.

Complexity increases when one works a growth model with more than one accumulation function. In order to illustrate this, we choose Lucas (1988). In his paper, Lucas (1988) uses the following human capital accumulation function, which is the source of endogenous growth in his model:

$$\dot{h} = a_h(1-u)h \quad (5)$$

Where h is human capital stock per person, a_h is productivity of education sector, and u is the share of human capital employed in final good production. Notably, there is no diminishing marginal returns to h in the accumulation function for generating endogenous growth. Suppose that social planner's problem is maximization of $U = \int_0^\infty e^{-(\rho-n)\cdot t} \cdot \frac{c^{1-\theta}-1}{1-\theta} dt$ subject to $\dot{K} = K^\alpha (uhL)^{1-\alpha} - c \cdot L$ and (1) and corresponding transversality conditions, where K is physical capital stock, L is labor stock and grows at rate n, and c is consumption per capita (all lower-case letters correspond to per capita versions of a variable). One may easily show that endogenous growth rate at steady state would be $g = \frac{1}{\theta}(a_h + n - \rho)$, and that $\hat{c}_{ss} = \hat{y}_{ss} = \hat{k}_{ss} = \hat{h}_{ss} = g$ (subscript ss indicates steady state). Instead, if the human capital accumulation function were defined with a counter-moving force, say, $\dot{h} = a_h(1-u)h - (n+\delta)h$, where δ is the decay rate of human capital, the endogenous growth rate at steady state would be $g = \frac{1}{\theta}(a_h - \delta - \rho)$. As long as $a_h > \delta + \rho$, there is no qualitative difference between the original and modified models. Heuristically speaking, this may be the reason why the counter-moving force in the accumulation function has been ignored in many endogenous growth models.

However, whenever there is diminishing-returns to the accumulation function, it is of considerable importance whether there is a counter-moving force or not,. For matter of illustration, let us continue with the Lucas (1988) model. Suppose that human capital accumulation function is $\dot{h} = a_h(1-u)h^\xi$ versus $\dot{h} = a_h(1-u)h^\xi - (n+\delta)\cdot h$. In the first case, in which there is decreasing returns to human capital accumulation but no counter force in the accumulation function, we find the model implies that the human capital accumulation sector (i.e. the education sector) will disappear in the long run from the model and that all existing human capital will be employed in private production.[1] Given that human capital does not depreciate, and that the returns to the education sector diminish, it is indeed optimal for the social planner (decentralized solution would not be different) to employ all human capital factors of production in final good production. In the second scenario, it is very simple to show that there is stationary state of h and u and other variables of the model. In particular, one can show that $Lim_{t\to\infty} h(t) = \left(\frac{a_h\cdot(1-u_{SS})}{n+\delta}\right)^{\frac{1}{1-\xi}}$ and that $Lim_{t\to\infty} u(t) = \frac{(\rho+\delta)-\xi(n+\delta)}{(\rho+\delta)-(1-\xi)(n+\delta)}$ ($u > 0$ implies $\xi > \frac{1}{2}$). Notably, the two scenarios are qualitatively different. This simple example shows clearly that accumulation functions should be defined, as required by theory.

Next, we would like to give one more example with more serious implications. Our example will be the seminal paper by Dasgupta and Heal (1974). Those familiar with this paper would know that the social planner's solution to their problem leads to the famous $\frac{\partial F_R}{\partial t}\frac{1}{F_R} = F_K$ equation, where R is the extraction quantity of a non-renewable resource, K is physical capital, F_M indicates marginal physical productivity for

[1] See Annex A for a formal proof.

$M = K, R$. This relationship leads to the differential equation $\dot{X} = X^{\alpha}$, where $X = \frac{K}{R}$ for the Cobb-Douglas technology. This differential equation behaves very much like (1). If there were depreciation in the model, however, the differential equation would be $\dot{X} = X^{\alpha} - \delta X$, which behaves very much like (3).

The reader would be entirely justified in asking what the significance of this is. However, the importance of defining the model as theory requires becomes clear whenever one looks at the decentralized solution of the same model. If the original Dasgupta and Heal (1974) model were modeled in a decentralized solution, the reader would see that $\frac{\partial F_R}{\partial t}\frac{1}{F_R} = F_K$ is nothing but the Hotelling's Rule: $\frac{\dot{q}}{q} = r$, where q is the price of non-renewable resource and r is the rental rate of capital.[2] If there is depreciation in the model, the equation becomes $\frac{\dot{q}}{q} = r - \delta$, where the right hand side is the real interest. In the first scenario, one finds q goes to infinity. In the second scenario, q converges to a constant value. Clearly, the two results are qualitatively different, and it must be true that one gives the wrong conclusions. Our point is that an accumulation function must be modeled as the theory requires. If there is, for example, physical capital in the model, then it must be subject to depreciation. In this case, therefore, the real interest rate and rental rate of capital must be different. In conclusion, to ensure accurate results, one should always follow the theory when defining an accumulation function, especially whenever there is no endogenous growth. Not to do this risks distorting the results.

3 Concluding Remarks

Differential equations are very sensitive to changes. Accumulation functions are no exception. It is important to define an accumulation function as theory requires. The general understanding in growth theory is that defining a diminishing marginal productivity is sufficient for generating steady state results. This perception however is not fully correct. If theory requires, one should always add a counter-moving force into the accumulation functions in order to ensure that the model has a steady state not only at growth rates but also at levels. Ignoring this fact risks distorting the results.

[2] Recall that Hotelling's rule is a non-arbitrage condition between non-renewable resource and financial assets, stating that the nonrenewable is also an asset and therefore its (real) price must grow at the real interest rate.

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Annex A
Solution of Lucas Model when $\dot{h} = a_h \cdot (1 - u) \cdot h^{\xi}$

Suppose that human capital education sector is defined as $\dot{h} = a_h \cdot (1 - u) \cdot h^{\xi}$, where a_h is productivity of education sector, $0 < \xi < 1$ and the rest of the model is same as Lucas (1988), except that we ignore externality for matter of simplicity. The social planner's solution of the model would imply the following Hamiltonian:

$$H = e^{-(\rho-n)\cdot t} \cdot \frac{C^{1-\theta}-1}{1-\theta} + \lambda \cdot \{Y - c \cdot L\} + \mu \cdot \{a_h \cdot (1-u) \cdot h^{\xi}\} \quad \text{(A.0)}$$

First order conditions would be as follows:

$$\frac{\partial H}{\partial c} = 0 \Rightarrow e^{-(\rho-n)\cdot t} \cdot c^{-\theta} + \lambda \cdot \{-L\} \quad \text{(A.1)}$$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow \lambda \cdot \left\{(1-\alpha)\frac{Y}{u}\right\} + \mu \cdot \{a_h \cdot (-1) \cdot h^{\xi}\} = 0 \quad \text{(A.2)}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial K} \Rightarrow \dot{\lambda} = -\lambda \cdot \left\{\alpha \frac{Y}{K}\right\} \quad \text{(4.3)}$$

$$\dot{\mu} = -\frac{\partial H}{\partial h} \Rightarrow \dot{\mu} = -\left[\lambda \cdot \left\{(1-\alpha)\frac{Y}{h}\right\} + \mu \cdot \{\xi \cdot a_h \cdot (1-u) \cdot h^{\xi-1}\}\right] \quad \text{(A.4)}$$

$$\dot{K} = \frac{\partial H}{\partial \lambda} \Rightarrow \dot{K} = K^{\alpha}(u \cdot h \cdot L)^{1-\alpha} - c \cdot L \quad \text{(4.5)}$$

$$\dot{h} = \frac{\partial H}{\partial \mu} \Rightarrow \dot{h} = a_h \cdot (1-u) \cdot h^{\xi} \quad \text{(A.6)}$$

(A.6) implies $\hat{h}_{ss} = a_h \cdot (1 - u_{ss}) \cdot h_{ss}^{\xi-1}$ at steady state, if there is one. Time derivative of both sides implies $\frac{d}{dt}(\hat{h}_{ss}) \equiv 0$ and $\frac{d}{dt}\left(a_h \cdot (1 - u_{ss}) \cdot h_{ss}^{\xi-1}\right) = 0 \Rightarrow \dot{u}_{ss} = (\xi - 1)(1 - u_{ss}) \cdot \frac{\dot{h}_{ss}}{h_{ss}}$. (A.2) yields $\lambda \cdot (1-\alpha)\frac{Y}{u} = \mu \cdot a_h \cdot h^{\xi}$. If we use this information in (A.4), we get $-\frac{\dot{\mu}}{\mu} = h^{\xi-1}[(1-\xi) \cdot a_h \cdot u + \xi \cdot a_h]$. Time derivative of this equation at steady state, if there is one, implies $[(1-\xi) \cdot u_{ss} + \xi] \cdot \frac{\dot{h}_{ss}}{h_{ss}} = \dot{u}_{ss}$. Using $\dot{u}_{ss} = (\xi - 1)(1 - u_{ss}) \cdot \frac{\dot{h}_{ss}}{h_{ss}}$ above implies,

$$[(1-\xi) \cdot u_{ss} + \xi] \cdot \frac{\dot{h}_{ss}}{h_{ss}} = (\xi - 1)(1 - u_{ss}) \cdot \frac{\dot{h}_{ss}}{h_{ss}} \Rightarrow \frac{\dot{h}_{ss}}{h_{ss}} = 0$$

Suppose that this is true. Then, $\frac{\dot{u}_{ss}}{u_{ss}} = 0$ must also hold. Under $\frac{\dot{h}_{ss}}{h_{ss}} = 0$, (A.6) implies $u_{ss} = 1$ or $h_{ss} = 0$. As $h_{ss} = 0$ is trivial, the model implies that education sector will disappear in the long run and all human capital will be employed in private production, that is, $u_{ss} = 1$. That would then imply $\hat{Y}_{ss} = \hat{K}_{ss} = n$ due to production function $Y = K^{\alpha}(uhL)^{1-\alpha}$. (4.5) implies $\hat{c}_{ss} = 0$ or $\hat{C}_{ss} = n$. (A.1) then implies $\hat{\lambda}_{ss} = -\rho$. (4.3) requires $\frac{Y_{ss}}{K_{ss}} = \frac{\rho}{\alpha}$. Log-differentiation of (A.2) implies $\hat{\mu}_{ss} = \hat{\lambda}_{ss} + n = -(\rho - n)$. This information used in (A.4) implies $h_{ss} = \left(\frac{a_h}{\rho-n}\right)^{\frac{1}{1-\xi}}$. Since boundedness from above of overall utility necessarily implies $\rho > n$, $h_{ss} > 0$. Using $-\hat{\lambda}_{ss} = \alpha \frac{Y_{ss}}{K_{ss}}$ from (4.3) in (4.5) yields $\frac{Y_{ss}}{K_{ss}} = \frac{C_{ss}}{K_{ss}}$. One can easily show that $k_{ss} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \left(\frac{a_h}{\rho-n}\right)^{\frac{1}{1-\xi}}$ from production function, where $k = \frac{K}{L}$. The rest follows straightforwardly.