Exercise VI
Two-Period Models—Partial Equilibrium

1. Suppose that Daniel has income of \( y_1 \) when he is young and \( y_2 \) when he is old. Initially, the real interest rate is \( r_1 \). The utility function of Daniel is

\[
U = \frac{c_1^{1-\theta}}{1-\theta} + \beta \frac{c_2^{1-\theta}}{1-\theta},
\]

where \( \beta \) is the discount factor.

(i) Find the optimal values of \( c_1 \) and \( c_2 \).

(ii) Show that \( \frac{\partial c_1^*}{\partial y_1} > 0 \), \( \frac{\partial c_2^*}{\partial y_1} > 0 \), and \( \frac{\partial s^*}{\partial y_1} > 0 \).

(iii) Show that \( \frac{\partial c_1^*}{\partial y_2} > 0 \), \( \frac{\partial c_2^*}{\partial y_2} > 0 \), and \( \frac{\partial s^*}{\partial y_2} < 0 \).

(iv) Show that \( \frac{\partial c_1^*}{\partial r} = \text{?} \), \( \frac{\partial c_2^*}{\partial r} > 0 \), and \( \frac{\partial s^*}{\partial r} = \text{?} \) if the consumer is a lender in the current period. (Difficult)

(v) Show that \( \frac{\partial c_1^*}{\partial r} < 0 \), \( \frac{\partial c_2^*}{\partial r} = \text{?} \), and \( \frac{\partial s^*}{\partial r} > 0 \) if the consumer is a borrower in the current period. (Difficult)

2. Suppose that Daniel has income of \( y_1 = 400 \) when he is young and \( y_2 = 100 \) when he is old. Initially, the real interest rate is \( r_1 = 25\% \). The utility function of Daniel is

\[
U = \frac{c_1^{1-\theta}}{1-\theta} + \beta \frac{c_2^{1-\theta}}{1-\theta},
\]

where \( \beta = 0.8 \) is the discount factor.

(i) Find the optimal values of \( c_1 \), \( c_2 \) and \( U(\cdot)U \) for \( \theta = 0.5 \).

(ii) Suppose now that \( y_1 \) has been raised to 500. Find the optimal values of \( c_1 \) and \( c_2 \).

(iii) Suppose now that \( y_2 \) has been raised to 150. Find the optimal values of \( c_1 \) and \( c_2 \).

(iv) Suppose now that \( r \) has been raised to 30\%. Find the optimal values of \( c_1 \) and \( c_2 \). Try to disentangle substitution effect from income effect, using the Hicksian compensation.
3. Use the following utility function and budget constraint to answer the given questions.

\[ U = \ln(c_1) + \beta \ln(c_2) \quad \text{s.t.} \quad c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \]

where \( c_1 \) is this period’s consumption and \( c_2 \) is next period’s consumption, \( y_1 \) is current income and \( y_2 \) is net period’s income, and \( r \) is the real interest rate.

(i) Determine the MRS (marginal rate of substitution) for this consumer.

(ii) Find the optimal consumption bundle \((c_1, c_2)\) as a function of \(y_1, y_2,\) and \(r\) for \(\beta = 1\). Interpret what \(\beta = 1\) does mean. Sketch a graph of this solution.

(iii) Let \(y_1 = y_2 = 105\) and \(r = .05\). Use these values to compute current consumption and determine if this consumer is a saver or borrower given this endowment. Show the endowment point on your graph.

(iv) For the same income endowment, determine what happens to \(c_1\) if \(r\) increases to \(r = 0.1\). Which effect dominates the current consumption decision: income or substitution effect?

4. Consider the following aggregate consumption function and government budget constraint:

\[ C_1 = \frac{1}{2} \left[ Y_1 - T_1 + \frac{Y_2 - T_2}{1+r} \right] \quad \text{and} \quad G_1 + \frac{G_2}{1+r} = T_1 + \frac{T_2}{1+r} \]

(i) Show that Ricardian Equivalence holds when consumers recognize that future taxes must cover the current deficit and future spending.

(ii) Since current consumption, \(C_1\), only changes if \(G_1\) or \(G_2\) changes, what happens to private savings \((Y-T-C)\) if current taxes are reduced, with no change in \(G\)? (You may assume \(Y\) is unchanged.)

(iii) Use the consumption function solved in terms of \(G_1\) and \(G_2\) to answer these questions numerically. Let \(Y_1 = Y_2 = 125\) and \(r = 0.05\).

- If \(G_1 = G_2 = 20\), what is current consumption, \(C_1\)?
- What happens to \(C_1\) if \(G_1\) decreases to 10 with no change in \(G_2\)?
- What happens to \(C_1\) if \(G_1\) decreases to 10 and \(G_2\) decreases to 9.5?
- Use your answers above to infer the MPC (marginal propensity to consume) for a temporary increase in income and the MPC for a permanent increase in income. (Hint: refer your textbook to answer this question)