Trade Rules for Uncleared Markets with a Variable Population*

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Abstract

We analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are possibly unequal. The agents have single peaked preferences on their consumption and production choices. For such markets, we analyze the implications of population changes as formalized by the well-known *consistency* and *population monotonicity* properties. We first characterize the subclass of "Uniform trade rules" that satisfies *Pareto optimality, no-envy, consistency*, and an informational simplicity property. Next, we characterize *trade rules* that satisfy *population monotonicity* together with *Pareto optimality, no-envy*, and *strategy-proofness* as well as *Pareto optimality, no-envy*, and *peak-only.*

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1 Introduction

We analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are possibly unequal. This stickiness of prices is observed in many markets, either because the price adjustment process is slow (such as in labor and housing markets) or because the price is controlled from the outside of the market (such as in health, education, and agriculture). There is a wide literature about this phenomenon. For a detailed discussion, see Benassy (1993, 1982, 2002).

In our model, buyers and sellers constitute two exogenously differentiated sets. There is only one traded commodity and sellers face demand from buyers. Buyers might be individuals or producers that use the commodity as input. We assume that the buyers have strictly convex preferences on consumption bundles. Thus, they have single-peaked preferences on the boundary of their budget sets, and therefore, on their consumption of the commodity. Similarly, we assume that the sellers have strictly convex production sets. Thus, their profits are single-peaked in their output.

We allow the population to be variable and analyze the implications of population changes.

A *trade rule* maps each economy to a feasible trade. In our model it is made up of two components: a *trade-volume rule* and an *allocation rule*. The *trade-volume rule* takes the preferences of the buyers and the sellers and a possible transfer and determines the trade-volume that will be carried out in the economy and thus, the total consumption and the total production. Then, the *allocation rule* allocates the total consumption among the buyers and the total production among the sellers.

Trade-volume rules are related to Moulin (1980) who analyzes the determination of a one-dimensional policy issue among agents with single-peaked preferences. Particularly, when there is only one buyer and one seller, the trade-volume is exactly like a public good for these two agents. However, this is no more true when there are multiple buyers and sellers.

The problem faced by the *allocation rule* is to allocate a social endowment (that is, the total consumption) among agents with single peaked preferences. This problem is extensively analyzed by the literature following Sprumont (1991). However, we also analyze markets with multiple buyers and sellers. Thus, our domain is an extension of Sprumont's domain[1]. Sprumont (1991) proposed and analyzed a "Uniform rule" which later became a central rule of that literature (for example,

[1] It coincides with the *just-buyer markets* (when there is no seller in the market) and the *just-seller markets* (when there is no buyer in the market) in our model.

see Dagan (1996), Ching (1992, 1994), Thomson (1994)). This rule will also play an important role in this paper.

Let us note that our model is not a simple conjunction of Moulin (1980) and Sprumont (1991). The interaction between the determination of the agent's shares and the trade-volume makes the model much richer. For example, the agents can manipulate their shares also by manipulating the trade-volume. Also, requirements like *Pareto optimality*, or "fairness" become much more demanding as what is to be allocated becomes endogenous. Another important difference is the existence of two types of agents (buyers and sellers) in our model. This duality limits the implications of requirements like *anonymity*, *no-envy*, and *population monotonicity*.

Our model is closely related to Kıbrıs and Küçükşenel (2009). They, however, analyze markets with a fixed population. These authors analyze a class of *trade rules* each of which is a composition of the Uniform rule with a *trade-volume rule* that picks the median of total demand, total supply and an exogenous constant. They show that this class uniquely satisfies *Pareto optimality, strategy proofness, no-envy,* and an informational simplicity axiom called *independence of trade-volume*.

Our model is also related to Thomson (1995) and Klaus, Peters, and Storcken (1997, 1998). They analyze the reallocation of an infinitely divisible commodity among agents with single peaked preferences and individual endowments. The agents whose endowments are greater than their peaks are considered as suppliers and those whose endowments are less than their peaks as demanders. These authors also characterize a *Uniform reallocation rule*. Note that, in their model the suppliers and the demanders are not differentiated. The identities of the agents depend on the relation between their peaks and endowments. Thus, a supplier by misrepresenting his preferences can turn into a demander. In our model, however, producers and consumers are exogenously distinct identities. This difference has important implications over the properties analyzed. For example, fairness properties are much weaker in our model since they only compare agents on the same side of the market. Also, in our model the agents do not have exogenously given endowments.

We introduce a class of *Uniform trade rules* each of which is a composition of the *Uniform rule* and a *trade-volume rule*. We axiomatically analyze *Uniform trade rules* on the basis of some central properties concerning variations of the population namely, *consistency* and *population monotonicity*. We also analyze the implications of standard properties such as *Pareto optimality, strategy-proofness*, and *no-envy*, and some informational simplicity properties such as *peak-only* and *strong independence of trade volume*.

Consistency has been analyzed in many contexts such as bargaining, coalitional form games, and taxation (for a detailed discussion, see Section 2). Loosely speaking, a rule is *consistent* if a recommendation it makes for an economy always agrees with its recommendations for the associated reduced economies obtained by the departure of some of the agents with their promised shares. *Consistency* is not well-defined for closed economies. Therefore, we analyze open economies and thus consider possible transfers to/from outside the economy. We show in Theorem 1 that a particular subclass of *Uniform trade rules* uniquely satisfies *consistency* together with *Pareto optimality, no-envy*, and *independence of trade-volume.*

Population monotonicity has also been widely analyzed in many different contexts such as in classical economies, single-peaked preferences, and public goods (for a detailed discussion, see Section 2). Loosely speaking, it requires that for a given economy, upon the departure (equivalently, arrival) of some agents, the welfare level of the remaining agents should be affected in the same direction. Since in our model, the agents on different sides of the market are exogenously differentiated, we analyze a *population monotonicity* property which only compares agents on the same side of the market. We first note that there are trade rules that simultaneously satisfy three properties, which are incompatible on Sprumont's domain: *Pareto optimality, no-envy*, and *population monotonicity.* In theorems 2 and 3, we characterize the subclass that additionally satisfies *strategy-proofness* and *peak-only*, respectively. We note that this subclass contains rules that do not always clear the short side[2] of the market. This might seem unrealistic at first glance. However, in real life, we can see examples of markets such that sometimes its long side is cleared, especially in markets with strong welfare implications for the society. For example, in health and education sectors, we can observe excess demand due to price regulations and thus overutilization of services such as overfilled schools and hospitals. Also, in many countries with excess labor supply, governments tend to over-employ in the public sector.

The paper is organized as follows. In Section 2, we introduce the model. In Section 3, we analyze the implications of *consistency* and in Section 4, the implications of *population monotonicity.*

2 Model

There are countably infinite universal sets, $\mathcal{B}$ of potential buyers and $\mathcal{S}$ of potential sellers. Let $\mathcal{B} \cap \mathcal{S} = \emptyset$. There is a perfectly divisible commodity that each seller produces and each buyer

[2]The *short side* of a market is where the total volume of desired transaction is smallest. It is thus the demand side if there is excess supply and the supply side if there is excess demand. The other side is called the *long side.*

consumes. Let $\mathbb{R}_+$ be the consumption/ production space for each agent. Let R be a preference relation over $\mathbb{R}_+$ and P be the strict preference relation associated with R. The preference relation R is **single-peaked** if there is $p(R) \in \mathbb{R}_+$ called the peak of R, such that for all $x, y \in \mathbb{R}_+$, $x < y \leq p(R)$ or $x > y \geq p(R)$ implies $y \; P \; x$. Each $i \in \mathcal{B} \cup \mathcal{S}$ is endowed with a continuous single-peaked preference relation R_i over $\mathbb{R}_+$. Let $\mathcal{R}$ denote the set of all continuous and single-peaked preference relations on $\mathbb{R}_+$.

Given a finite set $B \subset \mathcal{B}$ of buyers and a finite set $S \subset \mathcal{S}$ of sellers such that either $B \neq \emptyset$ or $S \neq \emptyset$, let $N = B \cup S$ be a **society**. Let $\mathcal{N}$ be the set of all societies. Let $\mathcal{N}_{\neq\emptyset}$ be the set of societies with a nonempty set of buyers and sellers. A preference profile R_N for a society N is a list $(R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}^N$ denote the set of all profiles for the society N. Given $N' \subset N$ and $R_N \in \mathcal{R}^N$, let $R_{N'} = (R_i)_{i \in N'}$ denote the restriction of R_N to N'.

A **market for society $\boldsymbol{N}$** $= B \cup S$ is a list $\boldsymbol{(R_B, R_S, T)}$ where $(R_B, R_S) \in \mathcal{R}^N$ is a profile of preferences for buyers and sellers and $T \in \mathbb{R}$ is a **transfer**. Note that T can both be positive and negative. A positive T represents a transfer made from outside. Thus, it is added to the production of the sellers and together they form the total supply. On the other hand, a negative T represents a transfer that must be made from the economy to the outside. Thus, it is considered as an addition to the total demand.

Given a market (R_B, R_S, T) for a society $N = (B \cup S)$, a **(feasible) trade** is a vector $z \in \mathbb{R}_+^{B \cup S}$ such that $\sum_B z_b = \sum_S z_s + T$. Let $Z(R_B, R_S, T)$ denote the set of all trades for (R_B, R_S, T).

There are two special subclasses of markets. A market (R_B, R_S, T) is a **just-buyer market** if $B \neq \emptyset$ and $S = \emptyset$. For such markets, the feasible trades are as follows. If $T \geq 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}_+^B : \; \sum_B z_b = T\}$. If $T < 0$, then $Z(R_B, R_S, T) = \emptyset$. (This is trivial because if there is no seller, all the agents are demanders, and thus, the supply is zero. Thus, if the outside transfer is positive, it would be equal to the total supply and it is divided among the buyers. However, if there is a negative transfer (that is, a transfer must be made to outside), since there is no seller, the transfer cannot be realized. Thus, in that case there is no trade.) A market (R_B, R_S, T) is a **just-seller market** if $B = \emptyset$ and $S \neq \emptyset$. For such markets, the feasible trades are as follows. If $T \leq 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}_+^S : \; \sum_S z_s + T = 0\}$. If $T > 0$, then $Z(R_B, R_S, T) = \emptyset$. (The explanation is similar to above.) Note that *just-buyer markets* and *just-seller markets* mathematically coincide with the allocation problems analyzed by Sprumont (1991). Thus, his domain is a restriction of ours.

Since the markets with no feasible trade are trivial, we restrict ourselves to the set of markets for which the set of trades is nonempty. Let $\mathcal{M}^N = \{(R_B, R_S, T) : (R_B, R_S) \in \mathcal{R}^N, T \in \mathbb{R}, \text{ and } Z(R_B, R_S, T) \neq \emptyset\}$ be the set all markets for society $N = B \cup S$ and let

$$\mathcal{M} = \bigcup_{N \in \mathcal{N}} \mathcal{M}^N$$

be the set of all markets. Let $\mathcal{M}_{\mathcal{B}} = \{(R_B, R_S, T) \in \mathcal{M} : B \neq \emptyset, S = \emptyset, \text{ and } T \geq 0\}$ be the set of *just-buyer markets* and $\mathcal{M}_{\mathcal{S}} = \{(R_B, R_S, T) \in \mathcal{M} : B = \emptyset, S \neq \emptyset, \text{ and } T \leq 0\}$ be the set of *just-seller markets*.

For the analysis of the properties, the following subclasses of markets turn out to be important. Let $\mathcal{M}^2 = \{(R_B, R_S, T) \in \mathcal{M} : |B| \geq 2 \text{ and } |S| \geq 2\}$ be the set of markets in which there are at least two buyers and two sellers. Let $\mathcal{M}^2_{\neq} = \{(R_B, R_S, T) \in \mathcal{M}^2 : \text{ there are } b_i, b_j \in B, \text{ and } s_k, s_l \in S \text{ such that } p(R_{b_i}) \neq p(R_{b_j}) \text{ and } p(R_{s_k}) \neq p(R_{s_l})\}$ be a subset of $\mathcal{M}^2$ that consists of markets with at least two buyers and two sellers with different peaks, respectively. Let $\mathcal{M}_0 = \{(R_B, R_S, T) \in \mathcal{M} : \text{ for } K \in \{B, S\}, \text{ for each } k \in K, p(R_k) = 0\}$ be the set of markets in which one side of the market consists of agents having a peak equal to 0. Also, let $\mathcal{M}_{nt} = \{(R_B, R_S, T) \in \mathcal{M} : T = 0\}$ be the set markets with no outside transfer. For notational simplicity, we will denote each $(R_B, R_S, T) \in \mathcal{M}_{nt}$ as (R_B, R_S).

Let $h(R_B, R_S, T)$ denote the **short side of the market $(\boldsymbol{R_B, R_S, T})$**, that is,

$$h(R_B, R_S, T) = \begin{cases} B & \text{if } \sum_B p(R_b) < \sum_S p(R_s) + T, \\ S & \text{if } \sum_S p(R_s) + T < \sum_B p(R_b). \end{cases}$$

A trade $z \in Z(R_B, R_S, T)$ is ***Pareto optimal with respect to $(\boldsymbol{R_B, R_S, T})$*** if there is no $z' \in Z(R_B, R_S, T)$ such that for all $i \in B \cup S$, $z'_i R_i z_i$ and for some $j \in B \cup S$, $z'_j P_j z_j$. The following lemma shows that in our framework, *Pareto optimality* is equivalent to the following three properties: (i) each agent in the short side of the market receives a share greater than or equal to his peak, (ii) each agent in the long side of the market receives a share less than or equal to his peak, and (iii) the total consumption is between the total supply and the total demand. Its proof is simple (see Kıbrıs and Küçükşenel (2009)).

Lemma 1 For each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$, a trade $z \in Z(R_B, R_S, T)$ is *Pareto optimal with respect to* (R_B, R_S, T) *if and only if for* $K \in \{B, S\}$, $h(R_B, R_S, T) = K$ *implies*

(i) *for each* $k \in K$, $p(R_k) \leq z_k$, ***(ii)*** *for each* $l \in N \setminus K$, $z_l \leq p(R_l)$, *and* ***(iii)*** $\sum_B z_b \in [\sum_B p(R_b), \sum_S p(R_s) + T]$[3].

A trade rule first determines the volume of trade that will be carried out in the economy and therefore, the total production and the total consumption. Then, it allocates the total production among the sellers and the total consumption among the buyers. Before defining a trade rule, we will first define a *trade-volume rule.*

A **trade-volume rule** $\Omega : \mathcal{M} \to \mathbb{R}^2_+$ associates each market (R_B, R_S, T) with a vector $\Omega(R_B, R_S, T) = (\Omega_B(R_B, R_S, T), \Omega_S(R_B, R_S, T))$ whose first coordinate, $\Omega_B(R_B, R_S, T)$ is the total consumption of the buyers and the second coordinate, $\Omega_S(R_B, R_S, T)$ is the total production of the sellers. Note that, for each market (R_B, R_S, T) and a *trade-volume rule* Ω, $\Omega_B(R_B, R_S, T) = \Omega_S(R_B, R_S, T) + T$. Thus, the volume of Ω_B determines the volume of Ω_S. Therefore, with an abuse of notation, we will sometimes call Ω_B a *trade-volume rule.*

In a *just-buyer market*, the transfer is divided among the buyers. Thus, the total consumption is equal to the transfer. In a *just-seller market*, however, the sellers produce an amount that corresponds to the transfer. Thus, in that case, the total production is equal to the absolute value of the transfer. Therefore, each *trade-volume rule* Ω satisfies the following:

$$\Omega(R_B, R_S, T) = \begin{cases} (T, 0) & \text{if } (R_B, R_S, T) \in \mathcal{M}_{\mathcal{B}} \\ (0, -T) & \text{if } (R_B, R_S, T) \in \mathcal{M}_{\mathcal{S}} \\ (\Omega_B(R_B, R_S, T), \Omega_S(R_B, R_S, T)) & \text{otherwise} \end{cases}$$

Note that, the trade-volume is fixed for the *just-buyer* and the *just-seller markets.* Thus, for simplicity, we will define a *trade-volume rule* only by the volume it chooses for the other markets.

Let $\mathcal{V}$ be the set of all *trade-volume rules.* Let $\mathcal{V}^{[short,long]}$ be the set of *trade-volume rules*, Ω each of which chooses a trade-volume between the total demand and supply of the market, that is, for each market (R_B, R_S, T),

$$\Omega(R_B, R_S, T) \in [\sum_B p(R_b), \sum_S p(R_s) + T].$$

[3]By $\sum_B z_b \in [\sum_B p(R_b), \sum_S p(R_s) + T]$, we mean the total consumption is between the total supply and the total demand, that is if $h(R_B, R_S, T) = S$, then consider $[\sum_S p(R_S) + T, \sum_B p(R_b)]$. In the rest of the paper, for simplicity we will sometimes use an interval notation in a similar meaning.

The following subclass of $\mathcal{V}^{[short,long]}$ will be used extensively in rest of the paper. Let $\mathcal{V}^{\{short,long\}}$ be the set of *trade-volume rules*, Ω each of which alternates between picking the total demand/supply of the short and the long side of the market, that is, for each market (R_B, R_S, T),

$$\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}.$$

Particularly, the following member of $\mathcal{V}^{\{short,long\}}$ will be important in our analysis: the *trade-volume rule* that always coincides with the total demand/supply of the long side[4] (Ω^{long}), that is,

$$\Omega^{long}(R_B, R_S, T) = \begin{cases} \sum_S p(R_s) + T & \text{if } h(R_B, R_S, T) = B \\ \sum_B p(R_b) & \text{if } h(R_B, R_S, T) = S \end{cases}$$

For a given market $(R_B, R_S, T) \in \mathcal{M}$ and $K \in \{B, S\}$, we say that a trade-volume rule $\boldsymbol{\Omega}$ **favors $\boldsymbol{K}$ in $\boldsymbol{(R_B, R_S, T)}$** if

$$\Omega(R_B, R_S, T) = \begin{cases} \sum_B p(R_b) & \text{if } K = B \\ \sum_S p(R_s) + T & \text{if } K = S \end{cases}$$

An **allocation rule** $f : \mathcal{M}_\mathcal{B} \cup \mathcal{M}_\mathcal{S} \rightarrow \cup_{M \in \mathcal{M}_\mathcal{B} \cup \mathcal{M}_\mathcal{S}} Z(M)$ associates each *just-buyer* and *just-seller* market (R_K, T) for $K \in \{B, S\}$, with a trade $z \in Z(R_K, T)$. For example, Uniform rule, U, introduced by Sprumont (1991) is very central in the literature. In our paper, also, it will be used extensively. Formally, it is defined as follows: for each $K \in \{B, S\}$, $(R_K, T) \in \mathcal{M}_\mathcal{K}$, and $k \in K$,

$$U_k(R_K, T) = \begin{cases} \min\{p(R_k), \lambda\} & \text{if } \sum_K p(R_k) \geq T \\ \max\{p(R_k), \mu\} & \text{if } \sum_K p(R_k) \leq T \end{cases}$$

where λ and μ is uniquely determined by the equations, $\sum_K \min\{p(R_k), \lambda\} = T$ and $\sum_K \max\{p(R_k), \mu\} = T$.

[4]The *trade-volume rule* that always coincides with the total demand/supply of the short side, Ω^{short} can be defined similarly. However, for the markets, (R_B, R_S, T) such that $h(R_B, R_S, T) = B$ and $\sum_B p(R_b) < T$, $\Omega^{short}_S(R_B, R_S, T) < 0$. Similarly, for the markets, (R_B, R_S, T) such that $h(R_B, R_S, T) = S$ and $\sum_S p(R_s) + T < 0$, $\Omega^{short}_B(R_B, R_S, T) < 0$. Thus, in our framework, Ω^{short} is not well-defined.

A **trade rule** $F : \mathcal{M} \to \cup_{M \in \mathcal{M}} Z(M)$ is a composition of a trade-volume rule Ω and an allocation rule f: $F = f \circ \Omega$. More precisely, for each market (R_B, R_S, T) and $K \in \{B, S\}$, $F_K(R_B, R_S, T) = f(R_K, \Omega_K(R_B, R_S, T))$. A *trade rule*, $F = U \circ \Omega$, that is composed of the *Uniform rule* and a *trade-volume rule* Ω is called the **uniform trade rule with respect to Ω**. In our analysis, $U \circ \Omega$ for some $\Omega \in \mathcal{V}^{\{short,long\}}$ turns out to be central. Kıbrıs and Küçükşenel (2009) characterize a particular class of *Uniform trade rules* for which Ω is the median of total demand, total supply, and an exogenous constant.

Let $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$ and F be a trade rule. Let $US^F(R_B, R_S, T) = \{i \in B \cup S : F_i(R_B, R_S, T) \neq p(R_i)\}$ be the **unsatisfied agents in (R_B, R_S, T) with respect to F**. Note that, if $F = f \circ \Omega^{long}$, then for each market $(R_B, R_S, T) \in \mathcal{M}$, $US^F(R_B, R_S, T) = h(R_B, R_S, T)$. Otherwise, however, there is a market $(R_B, R_S, T) \in \mathcal{M}$ such that $US^F(R_B, R_S, T) \cap B \neq \emptyset$, $US^F(R_B, R_S, T) \cap S \neq \emptyset$.

Now, we introduce properties of a trade rule. We start with efficiency. A trade rule F is **Pareto optimal** if for each $(R_B, R_S, T) \in \mathcal{M}$, the trade $F(R_B, R_S, T)$ is *Pareto optimal with respect to* (R_B, R_S, T).

Now, we present a fairness property. A trade is *envy free* if each buyer (respectively, seller) prefers his own consumption (respectively, production) to that of every other buyer (respectively, seller). A trade rule satisfies **no-envy** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $K \in \{B, S\}$, and $i, j \in K$, $F_i(R_B, R_S, T) R_i F_j(R_B, R_S, T)$. Since in our model the agents on different sides of the market are exogenously differentiated, this property only compares agents on the same side of the market.

The following is a property on nonmanipulability. It requires that regardless of the others' preferences, an agent is best-off with the trade associated with his true preferences. Formally, a trade rule F is **strategy proof** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, $F_i(R_i, R_{N \setminus i}, T)\ R_i\ F_i(R'_i, R_{N \setminus i}, T)$.

Next, we present some properties concerning possible variations in the number of agents. The first one is an adaptation of the standard *consistency* property to our domain. This property has been analyzed extensively in the context of bargaining by Lensberg (1987), single-peaked preferences by Thomson (1994), coalitional form games by Peleg (1986) and Hart and Mas-Colell (1989), taxation by Aumann and Maschler (1985) and Young (1987), cost allocation by Moulin (1985), fair allocation in classical economics by Thomson (1988), and matching by Sasaki and Toda (1992). To explain *consistency*, consider a trade rule F and a market (R_B, R_S, T). Suppose that F chooses the trade z. Imagine that some buyers and sellers leave with their shares they have been already assigned. This leads to a reduced problem in which the remaining agents, $(B' \cup S')$ are now facing an updated transfer from T to $T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s$. This practice is similar to the analysis of *consistency* in economies with individual endowments. Thomson (1992) introduced a "generalized economy" that consists of a preference profile of the agents, an endowment profile, and a trade vector that is updated in the reduced economies. The trade vector in that model corresponds in our model to the transfer. Consistency is about how the remaining agents' shares should be affected in the reduced problem. If F is consistent, it should assign to them the same shares as in the initial market. Without a transfer from outside, the recommendation for an economy may not be feasible for its reduced economies. This is one reason we consider open economies. Formally, given a trade rule F, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, a **reduced problem of $\boldsymbol{(R_B, R_S, T)}$ for $\boldsymbol{N'}$ at $\boldsymbol{z \equiv F(R_B, R_S, T)}$** is $r^z_{N'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s)$. A trade rule F is **consistent** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, if $z = F(R_B, R_S, T)$, then $z_{N'} = F(r^z_{N'}(R_B, R_S, T))$.

Consistency can also be defined for the *trade-volume rules* in a similar way. A *trade-volume rule* Ω is *consistent* if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$, and $z \in Z(R_B, R_S, T)$, $\Omega(r^z_{N'}(R_B, R_S, T)) = \sum_{B'} z_{b'}$.

Next is a standard *population monotonicity* property. It has been extensively analyzed in classical economies by Chichilnisky and Thomson (1987), Thomson (1987), Chun and Thomson (1988), Moulin (1992), and Chun (1986), on domains of economies with indivisible goods by Alkan (1989), Tadenuma and Thomson (1990, 1993), Moulin (1990), Bevia (1992), and Fleurbaey (1993), on domains of economies with both private and public goods by Thomson (1987), Moulin (1990), in single-peaked preferences by Thomson (1995), and Klaus (2001). *Population monotonicity* requires that upon the arrival (equivalently, departure) of some agents, the welfare levels of all remaining buyers and sellers should be affected in the same direction. Since in our model agents on different sides of the market are exogenously differentiated, *population monotonicity* only compares agents on the same side of the market. Formally, a trade rule F is **population monotonic** if for each $(B \cup S) \in \mathcal{N}_{\neq}$, $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$, $(B' \cup S') \supseteq (B \cup S)$, $K \in \{B', S'\}$, either (i) for each $i \in K$, $F_i(R_{B'}, R_{S'}, T)\ R_i\ F_i(R_B, R_S, T)$, or (ii) for each $i \in K$, $F_i(R_B, R_S, T)\ R_i\ F_i(R_{B'}, R_{S'}, T)$[5].

[5] *Population monotonicity* only analyzes societies with a nonempty set of buyers and sellers. If we include *just-buyer markets* and *just-seller markets*, by Thomson (1995) there is no trade rule that satisfies *Pareto optimality,*

There are two types of population expansions[6]. For a given market, by the arrival of some agents, the short side of the market may remain the same. We call such expansion as *simple population expansion.* Alternatively, the arrival of sufficiently many sellers (buyers) may turn an economy in which the short side is the buyers (sellers) into one in which the short side is the sellers (buyers). We call such expansion as the *radical population reduction.* Formally, they are defined as follows: let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$. Then, the set of **simple population expansions of $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$** is defined as $r^{sim}(R_B, R_S, T) = \{(B' \cup S') \supseteq (B \cup S) :$ $h(R_B, R_S, T) = K$ and $h(R_{B'}, R_{S'}, T) = K'$ for $K \in \{B, S\}\}$ and the set of **radical population expansions of $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$** is defined as $r^{rad}(R_B, R_S, T) = \{(B' \cup S') \supseteq (B \cup S) :\ h(R_B, R_S, T) =$ K and $h(R_{B'}, R_{S'}, T) = L'$ for $K \in \{B, S\}$ and $L = N \setminus K\}$.

Lastly, we present the following informational simplicity properties. *Peak-onliness* requires the trade only to depend on agents' peaks but not on the whole preference relation. According to this property, each agent may change his preference relation without changing his peak and this does not change the trade. Since obtaining the agents' whole preference relation is very difficult in real life, it is a requirement of informational simplicity. It is thus closely related to *strategy-proofness.* Formally, a trade rule F satisfies **peak-only** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T), (R'_B, R'_S, T) \in \mathcal{M}^N$, $p(R) = p(R')$ implies $F(R_B, R_S, T) = F(R'_B, R'_S, T)$.

Peak-onliness can also be defined for the *trade-volume rules*. *Peak-only-volume* requires the trade-volume only to depend on agents' peaks but not on the whole preference relation. Formally, a trade-volume rule, Ω satisfies **peak-only-volume** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T), (R'_B, R'_S, T) \in$ $\mathcal{M}^N$, $p(R) = p(R')$ implies $\Omega(R_B, R_S, T) = \Omega(R'_B, R'_S, T)$.

The following is also a simplicity property concerning the *trade-volume rule. Strong independence of trade-volume* requires the *trade-volume rule* only to depend on the total demand and supply but not on their individual components and the agents' identities. This property is a stronger version of *independence of trade volume* introduced by Kıbrıs et al (2009). *Independence of trade volume* relates two problems with the same set of agents. Formally, Ω satisfies **independence of trade volume** if for each $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T), (R'_B, R'_S, T) \in \mathcal{M}^N$, $\sum_{b \in B} p(R_b) = \sum_{b \in B} p(R'_b)$ and $\sum_{s \in S} p(R_s) = \sum_{s \in S} p(R'_s)$ implies $\Omega(R_B, R_S, T) = \Omega(R'_B, R'_S, T)$.

no-envy, and *population monotonicity.*

[6] *Population monotonicity* allows both type of expansions

3 Results

3.1 Consistency

The following lemma shows that for *Pareto optimal* rules, the reduction of a market does not change the short side (for its proof, please see the Appendix).

Lemma 2 *Let F be a Pareto optimal trade rule. Then, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$ with $B' \neq \emptyset$ and $S' \neq \emptyset$, if $z = F(R_B, R_S, T)$, then we have*

(i) $h(R_B, R_S, T) = B$ implies $h(r^z_{B' \cup S'}(R_B, R_S, T)) = B'$, and

(ii) $h(R_B, R_S, T) = S$ implies $h(r^z_{B' \cup S'}(R_B, R_S, T)) = S'$.

The following lemma analyzes the relationship between the properties satisfied by a trade rule $F = f \circ \Omega$, and its component f. It shows that *Pareto optimality, no-envy, peak-only, strategy proofness*, and *consistency* satisfied by F passes on to f (for its proof, please see the Appendix).

Lemma 3 *If a trade rule $F = f \circ \Omega$ satisfies one of the following properties, then f also satisfies that property: Pareto optimality, no-envy, peak-only, strategy proofness and consistency.*

Theorem 1 says that under the assumption of *independence of trade volume*, the subclass of *Uniform trade rules* $F = U \circ \Omega$ where Ω is consistent and $\Omega \in \mathcal{V}^{\{short,long\}}$ uniquely satisfies *Pareto optimality, no-envy* and *consistency* (for its proof, please see the Appendix):

Theorem 1 *Let $\Omega \in \mathcal{V}$ satisfy independence of trade-volume. A trade rule $F = f \circ \Omega$ satisfies Pareto optimality, no-envy, and consistency if and only if $f = U$ and Ω satisfies the following:*

(1.1) Ω is *consistent*,

(1.2) $\Omega \in \mathcal{V}^{\{short,long\}}$.

To prove Theorem 1, we use the following results. Dagan (1996) proves that the *Uniform rule* is the only *allocation rule* satisfying *Pareto optimality*, *no-envy*, and *bilateral consistency*, a weaker property than *consistency*. For its proof, please see Dagan (1996).

Lemma 4 *(Dagan, 1996) If the potential number of agents is at least 4 and if an economy consists of at least 2 agents, then f satisfies Pareto optimality, no-envy, and bilateral-consistency if and only if $f = U$.*

3.2 Population Monotonicity

Our first observation is that there are *trade rules* that simultaneously satisfy three properties which, on Sprumont's domain, are incompatible: *Pareto optimality, no-envy,* and *population monotonicity* (see Thomson (1995) for a discussion). The following lemma shows that the *Uniform trade rules* with respect to Ω^{long} satisfies *population monotonicity* together with *Pareto optimality* and *no-envy* (for its proof, please see the Appendix).

Lemma 5 The Uniform trade rule, $U \circ \Omega^{long}$ satisfies *Pareto optimality, no-envy, and population monotonicity.*

3.3 Peak-only

In this subsection, we analyze the implications of *population monotonicity* and *peak-only* together with some other properties.

In the following theorem, we characterize trade rules that satisfy *population monotonicity* together with *Pareto optimality, no-envy*, and *peak-only.* We show that each of these rules is a Uniform trade rule with respect to a trade volume rule, Ω that satisfies *peak-only* and also the following conditions: for each market, Ω chooses a trade volume that is between the total supply and the total demand. If, in addition, there are at least two buyers and two sellers with a different peaks, Ω either clears the short side or the long side of the market. Now, suppose for a given market there is a radical population expansion such that in the new bigger market there are at least one unsatisfied agent, i in the initial market and on the same side of him, at least one agent with a different peak. Then, in this new market, Ω should favor the side that i belongs to (for its proof, please see the Appendix).

Theorem 2 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, peak-only*, and *population monotonicity* if and only if $f = U$ and Ω satisfies the following:

(2.1) Ω is *peak-only*,

(2.2) $\Omega \in \mathcal{V}^{\{short,long\}}$ on $\mathcal{M}^2_{\neq}$ and $\Omega \in \mathcal{V}^{[short,long]}$ on $\mathcal{M} \setminus \mathcal{M}^2_{\neq}$,

(2.3) let $(R_B, R_S, T) \in \mathcal{M} \setminus \{\mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}}\}$ be such that there are $K \in \{B, S\}$, $i \in K \cap US^F(R_B, R_S, T)$, and $j \in K$ such that $p(R_i) \neq p(R_j)$. Then, for each $(B' \cup S') \in r^{rad}(R_B, R_S, T)$, Ω *favors* K' in $(R_{B'}, R_{S'}, T)$.

To prove Theorem 2, we need the following two lemmas. The first one says that an allocation rule satisfies *Pareto optimality*, *peak only*, and *no-envy* if and only if it coincides with the Uniform rule (for its proof, see Ching (1992)[7]).

Lemma 6 *(Ching, 1992)* An allocation rule, f satisfies *Pareto optimality, no-envy,* and *peak-only* if and only if $f = U$.

The following lemma shows that *peak-only* satisfied by a trade rule $F = f \circ \Omega$ passes on to f (its proof is similar to the proof of Lemma 3, thus we omit it).

Lemma 7 If a trade rule $F = f \circ \Omega$ satisfies *peak-only,* then f also satisfies *peak-only.*

Next, we add *independence of trade volume* to the list and analyze the implications of it with *population monotonicity* and *peak-only.* The following theorem shows that under the assumption of *independence of trade-volume*, the following subclass of *Uniform trade rules* is the only class containing rules that satisfy *population monotonicity* together with *Pareto optimality, no-envy,* and *peak only*: on $\mathcal{M} \setminus \{\mathcal{M}_0 \cup \mathcal{M}_{nt}\}$, Ω should always clear the long side of the market. On $\mathcal{M}_{nt}$, Ω should always clear either the short side of the market or the long side of the market. Also, on $\mathcal{M}_0$, Ω should choose a trade volume between the desired trade levels of the short and the long side of the market (for its proof, please see the Appendix).

Theorem 3 Let $\Omega \in \mathcal{V}$ satisfy *independence of trade-volume.* Then, a trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, peak-only*, and *population monotonicity* if and only if $f = U$ and Ω satisfies the following conditions:

(3.1) on $\mathcal{M}_0$, $\Omega \in \mathcal{V}^{[short,long]}$ and Ω is *peak-only*,

(3.2) on $\mathcal{M}_{nt} \setminus \mathcal{M}_0$, $\Omega = \Omega^{short}$ or $\Omega = \Omega^{long}$,

(3.3) on $\mathcal{M} \setminus \{\mathcal{M}_0 \cup \mathcal{M}_{nt}\}$, $\Omega = \Omega^{long}$.

3.4 Strategy proofness

In this subsection, we replace *peak only* with *strategy proofness* and analyze the implications of a misrepresentation of a preference relation.

[7] Ching (1992) shows that if an allocation rule satisfies *Pareto optimality*, *own-peak only*, and *no-envy*, then it coincides with the Uniform rule. *Own-peak only* is weaker than *peak-only.* Thus, Lemma 3 directly follows from Ching's result.

First, we characterize trade rules that satisfy *population monotonicity* together with *Pareto optimality, no-envy*, and *strategy proofness.* We show that each of these rules is a Uniform trade rule with respect to a trade volume rule, Ω that satisfies properties (2.2) and (2.3) of Theorem 2 and additionally the following property: let Ω clears the short (long) side of the market in (R_B, R_S, T). Then, *weak declaration invariance* requires that an agent from the long (short) side of the market can not change the trade volume by changing his preference relation unless he changes the short side. Formally, a trade volume rule Ω is **weak declaration invariant** if for each $(R_B, R_S, T) \in \mathcal{M}$, $K \in \{B, S\}$, if $h(R_B, R_S, T) = K$ and $\Omega(R_B, R_S, T) = \Omega^{short}(R_B, R_S, T)$ (respectively, $\Omega^{long}(R_B, R_S, T)$), then for each $i \in N \setminus K$ (respectively, $i \in K$), $R'_i \in \mathcal{R} \setminus \{R_i\}$ such that $h(R'_i, R_{N\setminus\{i\}}, T) = K$, $\Omega(R'_i, R_{N\setminus\{i\}}, T) = \Omega(R_B, R_S, T)$.

Theorem 4 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy-proofness*, and *population monotonicity* if and only if $f = U$ and the following conditions hold:

(4.1) Ω satisfies property (2.2) of Theorem 2,

(4.2) Ω satisfies property (2.3) of Theorem 2,

(4.3) Ω is *weak declaration invariant.*

We prove Theorem 4 with the help of the following three lemmas. The first one states that if a Uniform trade rule, $U \circ \Omega$ satisfies *Pareto optimality* and *strategy proofness*, then it satisfies a weaker *peak-only* property. This property requires the trade to remain the same when unsatisfied agents change their preference relations without changing their peaks (for its proof, see the Appendix).

Lemma 8 *If $F = U \circ \Omega$ satisfies* Pareto optimality *and* strategy proofness*, then for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = US^F(R_B, R_S, T)$, and $R'_{N'} \in \mathcal{R}^{N'}$ such that $p(R'_{N'}) = p(R_{N'})$, we have $F(R_B, R_S, T) = F(R'_{N'}, R_{N\setminus N'}, T)$.*

The following lemma shows that *strategy proofness* satisfied by a trade rule $F = f \circ \Omega$ passes on to f (its proof is similar to the proof of Lemma 3, thus we omit it).

Lemma 9 If a trade rule $F = f \circ \Omega$ is *strategy proof,* then f is also *strategy proof.*

The following lemma says that an allocation rule satisfies *Pareto optimality*, *no-envy*, and *strategy proofness* if and only if it coincides with the Uniform rule (for its proof, see Ching (1992).).

Lemma 10 *(Ching, 1992)* An allocation rule, f satisfies *Pareto optimality, no-envy,* and *strategy proofness* if and only if $f = U$.

Next, we add *independence of trade volume* to the list and analyze the implications of it with *population monotonicity* and *strategy-proofness*. We show that under the assumption of *independence of trade-volume*, a trade rule satisfies *population monotonicity* together with *Pareto optimality, no-envy*, and *strategy proofness* if and only if it is the Uniform trade rule with respect to Ω that satisfies the properties in Theorem 3 and additionally *weak-declaration invariance*(for its proof, please see the Appendix).

Theorem 5 Let $\Omega \in \mathcal{V}$ satisfy *independence of trade-volume*. Then, a trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy-proofness*, and *population monotonicity* if and only if $f = U$ and Ω satisfies the following:

(5.1) on $\mathcal{M}_0$, $\Omega \in \mathcal{V}^{[short,long]}$ and Ω is *weak-declaration invariant*,

(5.2) Ω satisfies property (3.2) of Theorem 3,

(5.3) Ω satisfies property (3.3) of Theorem 3.

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4 Appendix

Proof. (Lemma 3) Let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $(B' \cup S') \subseteq N$ be such that $B' \neq \emptyset$ and $S' \neq \emptyset$. Let $z \equiv F(R_B, R_S, T)$. First, suppose $h(R_B, R_S, T) = B$. Since F is *Pareto optimal*, z is *Pareto optimal with respect to* (R_B, R_S, T). Then, by Lemma 1 and 2, for each $b \in B$, $p(R_b) \leq z_b$ and for each $s \in S$, $z_s \leq p(R_s)$. Then,

$$\begin{aligned}\sum_{B \setminus B'} z_b + \sum_{B'} p(R_b) &\leq \sum_{B} z_b \\ &= \sum_{S} z_s + T \\ &\leq \sum_{S'} p(R_s) + \sum_{S \setminus S'} z_s + T.\end{aligned}$$

That is $\sum_{B'} p(R_b) \leq \sum_{S'} p(R_s) + T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s$. Note that $r^z_{B' \cup S'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T')$ for $T' = T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s$. Thus, $h(r^z_{B' \cup S'}(R_B, R_S, T)) = B'$. This proves (i). The proof of (ii) is similar. ■

Proof. (Lemma 4) First, suppose for a contradiction $F = f \circ \Omega$ satisfies *Pareto optimality* whereas f does not. Then, there is $K \in \{B, S\}$ and $(R_K, T) \in \mathcal{M}_{\mathcal{K}}$ such that $f(R_K, T)$ is not *Pareto optimal* with respect to (R_K, T). Then, since $(R_K, T) \in \mathcal{M}$ and $F(R_K, T) = f(R_K, T)$, $F(R_K, T)$ is not *Pareto optimal* with respect to (R_K, T), a contradiction to F being *Pareto optimal*. The other properties can be proved similarly.

■

Proof. (Theorem 1) The if part is straightforward and thus, omitted. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *consistency*, by Lemma 4, f also satisfies those properties. Then, by Lemma 5, $f = U$.

Now, let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$ and $(B' \cup S') \in \mathcal{N}$ be such that $(B' \cup S') \subseteq (B \cup S)$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(r^z_{B' \cup S'}(R_B, R_S, T))$. Since F is *consistent*, for each $i \in (B' \cup S')$, $z'_i = z_i$. Then, by the definition of Ω, $\Omega(r^z_{B' \cup S'}(R_B, R_S, T)) = \sum_{B'} z'_{b'} = \sum_{B'} z_{b'}$. Thus, Ω is *consistent*.

First, let $h(R_B, R_S, T) = S$. Suppose for a contradiction $\Omega(R_B, R_S, T) \notin \{\sum_B p(R_b), \sum_S p(R_s) + T\}$. Let $\sum_B p(R_b) = a$, $\sum_S p(R_s) + T = d$, and $\Omega(R_B, R_S, T) = c$. Since F is *Pareto optimal*, by

Lemma 1, $c \in (d,a)$. Let $\varepsilon \in \mathbb{R}_+$ be such that $\varepsilon < \min\{\frac{c}{n}, \frac{2(a-c)}{(n-2)}, \frac{2(n-1)(c-d)}{(m-1)(n-2)}\}$. Also let $m, n \in \mathbb{N}$ be such that $n \geq 3$ and $m > \max\{3, \frac{c-T}{d-T}\}$.

Let $(R_{B'}, R_{S'}, T) \in \mathcal{M}^{B' \cup S'}$ be such that $|B'| = n$, $|S'| = m$, and

$p(R_{b'_1}) = \frac{c}{n} - \varepsilon$, $p(R_{b'_2}) = \cdots = p(R_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1}$,
$p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)}$, $p(R_{s'_2}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} + \frac{(n-2)(m-3)\varepsilon}{2(n-1)(m-2)}$,
$p(R_{s'_3}) = \cdots = p(R_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{(n-2)\varepsilon}{(n-1)(m-2)}$.

Also, let $(R'_{B'}, R'_{S'}, T) \in \mathcal{M}^{B' \cup S'}$ be such that

$p(R'_{b'_1}) = \frac{c}{n} - \frac{\varepsilon}{2}$, $p(R'_{b'_2}) = \frac{a}{n-1} - \frac{c}{n(n-1)} - \frac{(n-3)\varepsilon}{2(n-1)}$, $p(R'_{b'_3}) = \cdots = p(R'_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1}$,
$p(R'_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}$, $p(R'_{s'_2}) = \cdots = p(R'_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{(n-2)\varepsilon}{(n-1)(m-2)}$.

Note that by the choice of ε and m, for each $k \in (B' \cup S')$, $p(R_{k'}) \geq 0$ and $p(R'_{k'}) \geq 0$. Also, $\sum_{B'} p(R_{b'}) = \sum_{B'} p(R'_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = \sum_{S'} p(R'_{s'}) = d - T$. Then, by *independence of trade volume*, $\Omega(R_{B'}, R_{S'}, T) = \Omega(R'_{B'}, R'_{S'}, T) = c$.

For each $K \in \{B', S'\}$, let $z_K \equiv F_K(R_{B'}, R_{S'}, T) = U(R_K, c)$ and $z'_K \equiv F_K(R'_{B'}, R'_{S'}, T) = U(R'_K, c)$. Since for each $i = 2, \cdots, n$, $p(R_{b'_1}) < \frac{c}{n} < p(R_{b'_i})$, $p(R'_{b'_1}) < \frac{c}{n} < p(R'_{b'_i})$, and $\frac{1}{(n-1)}(c - p(R'_{b'_1})) < p(R'_{b'_i})$, we have $z_{b'_1} = p(R_{b'_1}) = \frac{c}{n} - \varepsilon$, $z_{b'_i} = \frac{1}{n-1}(c - p(R_{b'_1})) = \frac{c}{n} + \frac{\varepsilon}{n-1}$, $z'_{b'_1} = p(R'_{b'_1}) = \frac{c}{n} - \frac{\varepsilon}{2}$, and $z'_{b'_i} = \frac{1}{n-1}(c - p(R'_{b'_1})) = \frac{c}{n} + \frac{\varepsilon}{2(n-1)}$.

Since for each $i = 2, \cdots, m$, $p(R_{s'_i}) < \frac{c-T}{m} < p(R_{b'_1})$, $p(R'_{s'_i}) < \frac{c-T}{m} < p(R'_{s'_1})$, and $\frac{1}{(m-1)}(c - T - p(R_{s'_1})) > p(R'_{s'_i})$, we have $z_{s'_1} = p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)}$, $z_{s'_i} = \frac{1}{m-1}(c - T - p(R_{s'_1})) = \frac{c}{m} - \frac{T}{m} - \frac{\varepsilon(n-2)}{2(m-2)(n-1)}$, $z'_{s'_1} = p(R'_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}$, and $z'_{s'_i} = \frac{1}{m-1}(c - T - p(R'_{s'_1})) = \frac{c}{m} - \frac{T}{m} - \frac{\varepsilon(n-2)}{(m-2)(n-1)}$.

Now, let $T' = \frac{2T}{m} + \frac{2(m-n)c}{mn} - \frac{3(n-2)\varepsilon}{2(n-1)}$ and consider the following two reduced problems:

(i) $r^z_{\{b'_1, b'_2, s'_1, s'_2\}}(R_{B'}, R_{S'}, T) = (R_{b'_1}, R_{b'_2}, R_{s'_1}, R_{s'_2}, T')$,
(ii) $r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T) = (R'_{b'_1}, R'_{b'_2}, R'_{s'_1}, R'_{s'_2}, T')$.

Note that, $p(R_{b'_1}) + p(R_{b'_2}) = p(R'_{b'_1}) + p(R'_{b'_2})$ and $p(R_{s'_1}) + p(R_{s'_2}) = p(R'_{s'_1}) + p(R'_{s'_2})$. Then, by *strong independence of trade volume*, $\Omega(r^z_{\{b'_1, b'_2, s'_1, s'_2\}}(R_{B'}, R_{S'}, T)) = \Omega(r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T))$. By *consistency*, for $i = 1, 2$, $F_{b'_i}(r^z_{b'_1, b'_2, s'_1, s'_2}(R_{B'}, R_{S'}, T)) = z_{b'_i}$ and $F_{b'_i}(r^{z'}_{b'_1, b'_2, s'_1, s'_2}(R'_{B'}, R'_{S'}, T)) = z'_{b'_i}$. Then,

$\Omega(r^z_{\{b'_1, b'_2, s'_1, s'_2\}}(R_{B'}, R_{S'}, T)) = z_{b'_1} + z_{b'_2} = \frac{2c}{n} + \frac{(2-n)\varepsilon}{n-1}$ and
$\Omega(r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T)) = z'_{b'_1} + z'_{b'_2} = \frac{2c}{n} + \frac{(2-n)\varepsilon}{2(n-1)}$.

Then, $\Omega(r^z_{\{b'_1, b'_2, s'_1, s'_m\}}(R_{B'}, R_{S'}, T)) \neq \Omega(r^{z'}_{\{b'_1, b'_2, s'_1, s'_2\}}(R'_{B'}, R'_{S'}, T))$, a contradiction. Thus, $\Omega(R_B, R_S, T) \in \{\sum_B p(b), \sum_S p(s) + T\}$. ■

Proof. (Lemma 6) Let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B\cup S}$. Without loss of generality, let $\sum_B p(R_b) \leq \sum_S p(R_s)+T$. Then, $\Omega^{long}(R_B, R_S, T) = \sum_S p(R_s)+T$. Let $z \equiv U \circ \Omega^{long}(R_B, R_S, T)$. Then, by the definition of U, for each $b \in B$, $z_b \geq p(R_b)$ and for each $s \in S$, $z_s = p(R_s)$. Let $(B' \cup S') \subseteq (B \cup S)$. Suppose first, $\sum_{B'} p(R_{b'}) \leq \sum_{S'} p(R_{s'}) + T$. Then, $\Omega^{long}(R_{B'}, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T$. Let $z' \equiv U \circ \Omega^{long}(R_{B'}, R_{S'}, T)$. Then, by the definition of U, for each $s' \in S'$, $z'_{s'} = p(R_{s'})$. Thus, for each $s' \in S'$, $z'_{s'}\ I_{s'}\ z_{s'}$.

Claim: We have either (i) for each $b' \in B'$, $p(R_{b'}) \leq z'_{b'} \leq z_{b'}$ or (ii) for each $b' \in B'$, $z'_{b'} \geq z_{b'}$.

Proof of Claim: Suppose for a contradiction there are $\tilde{b}, \bar{b} \in B'$ such that $z'_{\tilde{b}} > z_{\tilde{b}}$ and $z'_{\bar{b}} < z_{\bar{b}}$. By definition, $z'_{\tilde{b}} = \max\{\lambda', p(R_{\tilde{b}})\}$ where λ' is such that $\sum_{B'} \max\{\lambda', p(R_{b'})\} = \sum_{B'} p(R_{b'})$ and $z_{\tilde{b}} = \max\{\lambda, p(R_{\tilde{b}})\}$ where λ is such that $\sum_B \max\{\lambda, p(R_b)\} = \sum_B p(R_b)$. Since $z'_{\tilde{b}} > z_{\tilde{b}}$, $\lambda' > \lambda$. Then, $z'_{\bar{b}} = \max\{\lambda', p(R_{\bar{b}})\} \geq \max\{\lambda, p(R_{\bar{b}})\} = z_{\bar{b}}$, a contradiction to $z'_{\bar{b}} < z_{\bar{b}}$.

Thus, we have either (i) for each $b' \in B'$, $z'_{b'}\ R_{b'}\ z_{b'}$ or (ii) for each $b' \in B'$, $z_{b'}\ R_{b'}\ z'_{b'}$.

Now, suppose $\sum_{S'} p(R_{s'}) + T \leq \sum_{B'} p(R_{b'})$. Then, $\Omega^{long}(R_{B'}, R_{S'}, T) = \sum_{B'} p(R_{b'})$. Then, for each $b' \in B'$, $z'_{b'} = p(R_{b'})$. Thus, for each $b' \in B'$, $z'_{b'}\ R_{b'}\ z_{b'}$. Also, for each $s' \in S'$, $z'_{s'} \geq p(R_{s'})$, that is $z_{s'}\ R_{s'}\ z'_{s'}$. Therefore, $F = U \circ \Omega^{long}$ is *population monotonic.*

■

Proof. (Theorem 2) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *peak-only*, by the lemmas 3 and 7, f also satisfies those properties. Then, by Lemma 6, $f = U$. Since F is *peak-only*, Ω is also *peak-only* and this proves (2.1).

Since F is *Pareto optimal*, by Lemma 1, $\Omega \in \mathcal{V}^{[short,long]}$. Now, let $(R_B, R_S, T) \in \mathcal{M}^2_{>0}$. Let $|B| = n$ and $|S| = m$. Note that, $n \geq 2$ and $m \geq 2$. First, suppose $h(R_B, R_S, T) = S$. Since $\Omega \in \mathcal{V}^{[short,long]}$, $\Omega(R_B, R_S, T) \in [\sum_S p(R_s) + T, \sum_B p(R_b)]$. Suppose for a contradiction, $\Omega(R_B, R_S, T) \in (\sum_S p(R_s) + T, \sum_B p(R_b))$. Without loss of generality, enumerate $B = \{b_1, \cdots, b_n\}$ and $S = \{s_1, \cdots, s_m\}$ such that $p(R_{b_1}) \leq p(R_{b_2}) \leq \cdots \leq p(R_{b_n})$ and $p(R_{s_1}) \leq p(R_{s_2}) \leq \cdots \leq p(R_{s_m})$. Let $z \equiv F(R_B, R_S, T) = U \circ \Omega(R_B, R_S, T)$. Then, by the definition of U, $z_{b_1} \leq p(R_{b_1})$, $z_{b_n} < p(R_{b_n})$, $z_{s_1} > p(R_{s_1})$, and $z_{s_m} \geq p(R_{s_m})$. Now, let l be the smallest integer such that $p(R_{s_m})l + \sum_S p(R_s) + T > \sum_B p(R_b)$. Then, let $S' = S \cup \{s_{m+1}, \cdots, s_{m+l}\}$ and for each $i = 1, \cdots, l$, let $R_{s_{m+i}} = R_{s_m}$. Note that $h(R_B, R_{S'}, T) = B$. Let $z' \equiv F(R_B, R_{S'}, T)$. Since $\Omega \in \mathcal{V}^{[short,long]}$, $\Omega(R_B, R_{S'}, T) \in [\sum_B p(R_b), \sum_{S'} p(R_{s'}) + T]$. First, suppose $\sum_B p(R_b) < \Omega(R_B, R_{S'}, T) \leq \sum_{S'} p(R_{s'}) + T$. Then, by the definition of U, we have one of the following cases:

Case 1: Let $z_{b_1} = p(R_{b_1})$, $z_{b_n} < p(R_{b_n})$, and $z'_{b_1} = z'_{b_n} > p(R_{b_n})$. Then, let $R'_{b_n} \in \mathcal{R}$

be such that $p(R'_{b_n}) = p(R_{b_n})$ and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$. By *peak-only*, $F(R_{B\setminus\{b_n\}}, R'_{b_n}, R_S, T) = z$ and $F(R_{B\setminus\{b_n\}}, R'_{b_n}, R_{S'}, T) = z'$. Then, we have $z_{b_1}\ P_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Case 2: Let $z_{b_1} = p(R_{b_1})$, $z_{b_n} < p(R_{b_n})$, $z'_{b_1} > p(R_{b_1})$, and $z'_{b_n} = p(R_{b_n})$. Then, we have $z_{b_1}\ P_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Case 3: Let $z_{b_1} = z_{b_n} < p(R_{b_1})$, and $z'_{b_1} = z'_{b_n} > p(R_{b_n})$. Then, consider R'_{b_1} and R'_{b_n} such that $p(R'_{b_1}) = p(R_{b_1})$, $p(R'_{b_n}) = p(R_{b_n})$, $z_{b_1}\ P'_{b_1}\ z'_{b_1}$, and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$. By *peak-only,* $F(R_{B\setminus\{b_1,b_n\}}, R'_{b_1}, R'_{b_n}, R_S, T) = z$ and $F(R_{B\setminus\{b_1,b_n\}}, R'_{b_1}, R'_{b_n}, R_{S'}, T) = z'$. Then, we have $z_{b_1}\ P'_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P'_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Case 4: Let $z_{b_1} = z_{b_n} < p(R_{b_1})$, $z'_{b_1} > p(R_{b_1})$, and $z'_{b_n} = p(R_{b_n})$. Then, consider R'_{b_1} such that $p(R'_{b_1}) = p(R_{b_1})$ and $z_{b_1}\ P'_{b_1}\ z'_{b_1}$. By *peak-only,* $F(R_{B\setminus\{b_1\}}, R'_{b_1}, R_S, T) = z$ and $F(R_{B\setminus\{b_1\}}, R'_{b_1}, R_{S'}, T) = z'$. Then, we have $z_{b_1}\ P'_{b_1}\ z'_{b_1}$ and $z'_{b_n}\ P_{b_n}\ z_{b_n}$, a contradiction to F satisfying *population monotonicity.*

Second, suppose $\Omega(R_B, R_{S'}, T) = \sum_B p(R_b)$. Then, similar argument proves that in each case of z_{s_1}, z'_{s_1}, z_{s_m}, and z'_{s_m}, there is a violation of *population monotonicity.* Thus, $\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}$. Similar argument proves the other case in which $h(R_B, R_S, T) = B$, for this just replace S with B. This proves (2.2).

For (2.3), let $(R_B, R_S, T) \in \mathcal{M} \setminus \{\mathcal{M}_\mathcal{B} \cup \mathcal{M}_\mathcal{S}\}$ and $(B' \cup S') \in r^{rad}(R_B, R_S, T)$. Also, let $K' \in \{B', S'\}$ be such that there is $i \in K' \cap US^F(R_B, R_S, T)$ and there is $j \in K'$ such that $p(R_j) \neq p(R_i)$. First, let $h(R_B, R_S, T) = K$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv (R_{B'}, R_{S'}, T)$. By *Pareto optimality*, Lemma 1 implies $z_i > p(R_i)$ and $z_j \geq p(R_j)$. Since $(B' \cup S') \in r^{rad}(R_B, R_S, T)$, $h(R_{B'}, R_{S'}, T) = N' \setminus K' \equiv L'$. Suppose for a contradiction Ω does not favor K' in $(R_{B'}, R_{S'}, T)$. Then, by Lemma 1, for each $k' \in K'$, $z'_{k'} \leq p(R_{k'})$. Then, we have four cases:

Case 1: Let $z_i > p(R_i)$, $z_j = p(R_j)$, $z'_i = p(R_i)$, and $z'_j < p(R_j)$. Then, we have $z'_i\ P_i\ z_i$ and $z_j\ P_j\ z'_j$, a contradiction to F satisfying *population monotonicity.*

Case 2: Let $z_i > p(R_i)$, $z_j = p(R_j)$, $z'_i = z'_j < \min\{p(R_i), p(R_j)\}$. Then, consider $R'_i \in \mathcal{R}$ such that $p(R'_i) = p(R_i)$ and $z'_i\ P'_i\ z_i$. By *peak-only,* $F(R'_i, R_{(B\cup S)\setminus\{i\}}, T) = z$ and $F(R'_i, R_{(B'\cup S')\setminus\{i\}}, T) = z'$. Then, we have $z'_i\ P'_i\ z_i$ and $z_j\ P_j\ z'_j$, a contradiction to F satisfying *population monotonicity.*

Case 3: Let $z_i = z_j > \max\{p(R_i), p(R_j)\}$, $z'_i = p(R_i)$, and $z'_j < p(R_j)$. Then, consider $R'_j \in \mathcal{R}$ such that $p(R'_j) = p(R_j)$ and $z_j\ P'_j\ z'_j$. By *peak-only,* $F(R'_j, R_{(B\cup S)\setminus\{j\}}, T) = z$ and $F(R'_j, R_{(B'\cup S')\setminus\{j\}}, T) = z'$. Then, we have $z'_i\ P_i\ z_i$ and $z_j\ P'_j\ z'_j$, a contradiction to F satisfying *population monotonicity.*

Case 4: Let $z_i = z_j > \max\{p(R_i), p(R_j)\}$ and $z'_i = z'_j < \min\{p(R_i), p(R_j)\}$. Then, let $R'_i \in \mathcal{R}$ be such that $p(R'_i) = p(R_i)$ and $z_i \; P'_i \; z'_i$. Also, let $R'_j \in \mathcal{R}$ be such that $p(R'_j) = p(R_j)$ and $z'_j \; P'_j \; z_j$. By *peak-only*, $F(R'_i, R'_j, R_{(B\cup S)\setminus\{i,j\}}, T) = z$ and $F(R'_i, R'_j, R_{(B'\cup S')\setminus\{i,j\}}, T) = z'$. Then, we have $z_i \; P'_i \; z'_i$ and $z'_j \; P'_j \; z_j$, a contradiction to F satisfying *population monotonicity*.

Thus, if $h(R_B, R_S, T) = K$, Ω must favor K' in $(R_{B'}, R_{S'}, T)$. If $h(R_B, R_S, T) = N \setminus K$, the proof is very similar. Thus, Ω must favor K' in $(R_{B'}, R_{S'}, T)$ and this proves (2.3). ■

Proof. (Theorem 3) Let Ω satisfy *independence of trade volume*. Let $F = f \circ \Omega$ satisfy *Pareto optimality, no-envy, peak-only*, and *population monotonicity*. By Theorem 2, $f = U$ and $\Omega \in \mathcal{V}^{\{short,long\}}$ on $\mathcal{M}^2_{\neq}$, and $\Omega \in \mathcal{V}^{[short,long]}$ on $\mathcal{M} \setminus \mathcal{M}^2_{\neq}$. Then, by *independence of trade volume*, $\Omega \in \mathcal{V}^{\{short,long\}}$ on $\mathcal{M} \setminus \mathcal{M}_0$ and $\Omega \in \mathcal{V}^{[short,long]}$ on $\mathcal{M}_0$. This proves (3.1). We will prove (3.2) by the following two claims:

Claim 1. For each $(R_B, R_S), (R'_{B'}, R'_{S'}) \in \mathcal{M}_{nt}$ such that for $K \in \{B, S\}$, $h(R_B, R_S) = K$ and $h(R'_{B'}, R'_{S'}) = K'$, we have either $[\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{short}(R'_{B'}, R'_{S'})]$ or $[\Omega(R_B, R_S) = \Omega^{long}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{long}(R'_{B'}, R'_{S'})]$.

Proof of Claim 1. Without loss of generality, let $K = B$. Suppose for a contradiction, let $\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{long}(R'_{B'}, R'_{S'})$. Let $\sum_B p(R_b) = a$, $\sum_{B'} p(R'_{b'}) = a'$, $\sum_S p(R_s) = d$, and $\sum_{S'} p(R'_{s'}) = d'$. First, let $a > a'$ and $d > d'$. Now, let $\tilde{B} = \{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\}$ and $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3\}$. Also, let $(R_{\tilde{B}}, R_{\tilde{S}}) \in \mathcal{M}_{nt}$ be such that $p(R_{\tilde{b}_1}) = a'/3$, $p(R_{\tilde{b}_2}) = 2a'/3$, $p(R_{\tilde{b}_3}) = a - a'$, $p(R_{\tilde{s}_1}) = d'/3$, $p(R_{\tilde{s}_2}) = 2d'/3$, and $p(R_{\tilde{s}_3}) = d - d'$. Note that $\sum_{\tilde{B}} p(R_{\tilde{b}}) = a$ and $\sum_{\tilde{S}} p(R_{\tilde{s}}) = d$. Then, by *independence of trade volume*, $\Omega(R_{\tilde{B}}, R_{\tilde{S}}) = \Omega^{short}(R_{\tilde{B}}, R_{\tilde{S}}) = a$. Note that, there is $\tilde{s}_i \in \tilde{S}$ such that $\tilde{s}_i \in US^F(R_{\tilde{B}}, R_{\tilde{S}})$. Similarly, let $\tilde{B}' = \{\tilde{b}'_1, \tilde{b}'_2\}$ and $\tilde{S}' = \{\tilde{s}_1, \tilde{s}_2\}$. $(R_{\tilde{B}'}, R_{\tilde{S}'}) \in \mathcal{M}_{nt}$ be such that $p(R_{\tilde{b}'_1}) = a'/3$, $p(R_{\tilde{b}'_2}) = 2a'/3$, $p(R_{\tilde{s}'_1}) = d'/3$, $p(R_{\tilde{s}'_2}) = 2d'/3$. Note that $\sum_{\tilde{B}'} p(R_{\tilde{b}'}) = a'$ and $\sum_{\tilde{S}'} p(R_{\tilde{s}'}) = d'$. Then, by *independence of trade volume*, $\Omega(R_{\tilde{B}'}, R_{\tilde{S}'}) = \Omega^{long}(R_{\tilde{B}'}, R_{\tilde{S}'}) = d'$. Note that $\tilde{b}'_1 \in US^F(R_{\tilde{B}'}, R_{\tilde{S}'})$. Now, let $B'' = \tilde{B} \cup \{\tilde{b}_4\}$ and $S'' = \tilde{S}$. Let $R_{\tilde{b}_4} \in \mathcal{R}$ be such that $p(R_{\tilde{b}_4}) = d + d' - a$. Now, consider $(R_{B''}, R_{S''})$. Note that $\sum_{B''} p(R_{\tilde{b}}) = d + d'$ and $\sum_{S''} p(R_{\tilde{s}}) = d$. Thus, $h(R_{B''}, R_{S''}) = S''$. That is, $(B'' \cup S'') \in (r^{rad}(R_{\tilde{B}}, R_{\tilde{S}}) \cap r^{rad}(R_{\tilde{B}'}, R_{\tilde{S}'})$. Then, by (2.3) of Theorem 2, if we consider $(R_{\tilde{B}}, R_{\tilde{S}})$, Ω should favor S''. Also, by (2.3) of Theorem 2, if we consider $(R_{\tilde{B}'}, R_{\tilde{S}'})$, Ω should favor B'', a contradiction. Thus, we have either $[\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{short}(R'_{B'}, R'_{S'})]$ or $[\Omega(R_B, R_S) = \Omega^{long}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{long}(R'_{B'}, R'_{S'})]$. The proofs of the other relations of a, a', d, and d' are very similar.

Claim 2. On $\mathcal{M}_{nt}$, $\Omega = \Omega^{short}$ or $\Omega = \Omega^{long}$.

Proof of Claim 2. By Claim 1, first suppose for each $(R_B, R_S) \in \mathcal{M}_{nt}$ such that $h(R_B, R_S) = B$, $\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$. Let $(R_B, R_S) \in \mathcal{M}_{nt}$ be such that $h(R_B, R_S) = B$. Let $\sum_B p(R_b) = a$ and $\sum_S p(R_s) = d$. Then, $\Omega(R_B, R_S) = a$. Now, let $B' = \{b'_1, b'_2\}$ and $S' = \{s'_1, s'_2\}$. Let $R_{B'}, R_{S'} \in \mathcal{R}$ be such that $p(R_{b'_1}) = a/3$, $p(R_{b'_2}) = 2a/3$, $p(R_{s'_1}) = d/3$, and $p(R_{s'_2}) = 2d/3$. Note that, $\sum_{B'} p(R_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = d$. Then, by *independence of trade volume*, $\Omega(R_{B'}, R_{S'}) = a$. Then, $s'_2 \in US^F(R_{B'}, R_{S'})$. Now, let $B'' = B' \cup \{b'_3\}$ and $R_{b'_3} \in \mathcal{R}$ be such that $p(R_{b'_3}) = d$. Consider $(R_{B''}, R_{S'}) \in \mathcal{M}_{nt}$. Note that $\sum_{B''} p(R_{b''}) = a + d$. Thus, $h(R_{B''}, R_{S'}) = S'$, that is, $(B'' \cup S') \in r^{rad}(R_{B'}, R_{S'})$. Then, by Theorem 2, Ω should favor S' in $(R_{B''}, R_{S'})$, that is, $\Omega(R_{B''}, R_{S'}) = \Omega^{short}(R_{B''}, R_{S'})$. Then, by Claim 1, for each $(R_B, R_S) \in \mathcal{M}_{nt}$ such that $h(R_B, R_S) = S$, $\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$. The proof of the other case in which for each $(R_B, R_S) \in \mathcal{M}_{nt}$ such that $h(R_B, R_S) = B$, $\Omega(R_B, R_S) = \Omega^{long}(R_B, R_S)$ is similar. Thus, on $\mathcal{M}_{nt}$, $\Omega = \Omega^{short}$ or $\Omega = \Omega^{long}$.

The following Claims 3 and 4 prove (3.3).

Claim 3. For each $T > 0$, $(R_B, R_S, T) \in \mathcal{M} \setminus \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$, $\Omega(R_B, R_S, T) = \Omega^{long}(R_B, R_S, T)$.

Proof of Claim 3. Let $T > 0$. Let $(R_B, R_S, T) \in \mathcal{M} \setminus \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$. Let $\sum_B p(R_b) = a$ and $\sum_S p(R_s) = d$. First, suppose $h(R_B, R_S, T) = S$, that is $d + T < a$. Then, let $B' = \{b'_1, b'_2\}$ and $S' = \{s'_1, s'_2\}$. Let $R_{(B' \cup S)} \in \mathcal{R}^{B' \cup S'}$ be such that $p(R_{b'_1}) = T/6$, $p(R_{b'_2}) = T/3$, $p(R_{s'_1}) = d/3$, and $p(R_{s'_2}) = 2d/3$. Note that $(R_{B'}, R_{S'}, T) \in \mathcal{M} \setminus \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$. Then, $\Omega(R_{B'}, R_{S'}, T) \in \{\sum_{B'} p(R_{b'}), \sum_{S'} p(R_{s'}) + T\}$. Note that $\sum_{B'} p(R_{b'}) = T/2$ and $\sum_{S'} p(R_{s'}) + T = d + T$. Thus, $h(R_{B'}, R_{S'}, T) = B'$ and $\sum_{B'} p(R_{b'}) < T$. Then, by *feasibility*, $\Omega(R_{B'}, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T = \Omega^{long}(R_{B'}, R_{S'}, T)$. Then, $F_{b'_1}(R_{B'}, R_{S'}, T) > T/6$. Now, let $B'' = B' \cup \{b'_3\}$ and $R_{b'_3} \in \mathcal{R}$ be such that $p(R_{b'_3}) = a - T$. Note that $(R_{B''}, R_{S'}, T) \in \mathcal{M} \setminus \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$ and $\sum_{B''} p(R_{b'}) = a$, and $\sum_{S'} p(R_{s'}) = d$. Then, by Theorem 2, Ω should favor B'' in $(R_{B''}, R_{S'}, T)$, that is $\Omega(R_{B''}, R_{S'}, T) = \Omega^{long}(R_{B''}, R_{S'}, T)$. Then, by *independence of trade volume*, $\Omega(R_B, R_S, T) = \Omega^{long}(R_B, R_S, T)$. Now, suppose $h(R_B, R_S, T) = B$. By Theorem 2, $\Omega(R_B, R_S, T) \in \{\Omega^{short}(R_B, R_S, T), \Omega^{long}(R_B, R_S, T)\}$. Suppose for a contradiction $\Omega(R_B, R_S, T) = \Omega^{short}(R_B, R_S, T)$. let $B' = \{b'_1, b'_2\}$ and $S' = \{s'_1, s'_2\}$. Let $R_{(B' \cup S)} \in \mathcal{R}^{B' \cup S'}$ be such that $p(R_{b'_1}) = a/3$, $p(R_{b'_2}) = 2a/3$, $p(R_{s'_1}) = d/3$, and $p(R_{s'_2}) = 2d/3$. Note that $\sum_{B'} p(R_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = d$. Then, by *independence of trade volume*, $\Omega(R_{B'}, R_{S'}, T) = \Omega(R_B, R_S, T) = \Omega^{short}(R_B, R_S, T)$. Then, $s'_2 \in US^F(R_{B'}, R_{S'}, T)$. Now, let $B'' = B' \cup \{b'_3\}$ and $R_{b'_3} \in \mathcal{R}$ be such that $p(R_{b'_3}) = d + T$. Note that $(R_{B''}, R_{S'}, T) \in \mathcal{M} \setminus \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$ and $\sum_{B''} p(R_{b'}) = a + d + T$, and $\sum_{S'} p(R_{s'}) = d$. Note that, $h(R_{B''}, R_{S'}, T) = S$. Then, $(B'' \cup S'') \in r^{rad}(R_{B'}, R_{S'}, T)$ and by Theorem 2, Ω should favor S' in $(R_{B''}, R_{S'}, T)$. Thus, $\Omega(R_{B''}, R_{S'}, T) = \Omega^{short}(R_{B''}, R_{S'}, T)$, a contradiction to the first case.

Claim 4. For each $T < 0$, $(R_B, R_S, T) \in \mathcal{M} \backslash \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$, $\Omega(R_B, R_S, T) = \Omega^{long}(R_B, R_S, T)$.

Proof of Claim 4. The proof is very similar to the proof of Claim 3.

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Proof. (Lemma 8) Let $U \circ \Omega$ satisfy *Pareto optimality* and *strategy proofness*. Let $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$. Let $N' \equiv US^F(R_B, R_S, T)$ and $R'_{N'} \in \mathcal{R}^{N'}$ be such that $p(R'_{N'}) = p(R_{N'})$. Let $K \in \{B, S\}$ and let $i \in N' \cap K$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(R'_i, R_{N \backslash \{i\}}, T)$. Without loss of generality, let $z_i > p(R_i)$.

Claim 1. ($z'_i = z_i$) Suppose not. Since $p(R_i) = p(R'_i)$, $h(R'_i, R_{N\backslash\{i\}}, T) = h(R_N, T)$. Then, by Lemma 1, $z'_i \geq p(R_i) = p(R'_i)$. If $z'_i > z_i$, then when the true preference relation of i is R'_i, he manipulates z' by pretending as if his true preference relation is R_i. Similarly, if $z_i > z'_i$, then when the true preference relation of i is R_i, he manipulates z by pretending as if his true preference relation is R'_i. Thus, $z'_i = z_i$.

Claim 2. ($\Omega(R_B, R_S, T) = \Omega(R'_i, R_{B\backslash\{i\}}, R_S, T)$) Since $z_i > p(R_i)$, by the definition of U, $z_i = \max\{\lambda, p(R_i)\} = \lambda$ where λ satisfies $\sum_K \max\{\lambda, p(R_k)\} = \Omega(R_B, R_S, T)$. Similarly, $z'_i = \max\{\lambda', p(R_i)\}$ where λ' satisfies $\sum_K \max\{\lambda', p(R_k)\} = \Omega(R'_i, R_{N\backslash i}, T)$. By Claim 1, $z'_i = z_i = \lambda \neq p(R_i)$. Then, $z'_i = \lambda'$ and so $\lambda' = \lambda$. Thus, $\Omega(R_B, R_S, T) = \Omega(R'_i, R_{N\backslash\{i\}}, T)$.

Claim 3. (for each $j \in N \setminus \{i\}$, $z'_j = z_j$) It follows from $F = U \circ \Omega$ and Claim 2.

By Claim 1 and 3, $z' = z$. Now, let $j \in N' \backslash \{i\}$ and apply the same argument to $(R'_i, R_{N\backslash\{i\}}, T)$. Repeating the similar argument to each $k \in N'$ proves that $U \circ \Omega(R'_{N'}, R_{N\backslash N'}, T) = U \circ \Omega(R_B, R_S, T)$.

■

Proof. (Theorem 4) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *strategy proofness*, by the lemmas 3 and 7, f also satisfies those properties. Then, by Lemma 10, $f = U$.

The proofs of (4.1) and (4.2) are the same as the proof of (2.2) and (2.3), respectively. The only difference is that we use Lemma 8 instead of *peak-only*.

For (4.3), let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$. For $(II.1)$, let $\Omega(R_B, R_S, T) = \sum_B p(R_b)$. Let $s' \in S$ be such that $U \circ \Omega_{s'}(R_B, R_S, T) \neq p(R_{s'})$. Let $R'_{s'} \in \mathcal{R} \setminus \{R_{s'}\}$ be such that

$h(R_B, R_{S\setminus\{s'\}}, R'_{s'}, T) = h(R_B, R_S, T)$. Without loss of generality, let $h(R_B, R_S, T) = B$. Let $z \equiv U \circ \Omega(R_B, R_S, T)$ and $z' \equiv U \circ \Omega(R_B, R_{S\setminus\{s'\}}, R'_{s'}, T)$. Suppose for a contradiction, $\Omega(R_B, R_{S\setminus\{s'\}}, R'_{s'}, T) = \sum_{S\setminus\{s'\}} p(R_s) + p(R'_{s'}) + T$. By *Pareto optimality*, $z_{s'} < p(R_{s'})$ and $z'_{s'} = p(R'_{s'})$. Since $h(R_B, R_{S\setminus\{s'\}}, R'_{s'}, T) = h(R_B, R_S, T)$, $p(R'_{s'}) \geq z_{s'}$. If $p(R'_{s'}) < p(R_{s'})$, then when the true preference relation of s' is $R_{s'}$, he pretends as if his preference relation is $R'_{s'}$, a contradiction to *strategy proofness*. If $p(R'_{s'}) > p(R_{s'})$, then consider $R''_{s'} \in \mathcal{R}$ such that $p(R''_{s'}) = p(R_{s'})$ and $z'_{s'} P''_{s'} z_{s'}$. By Lemma 8, $F(R_B, R''_{s'}, R_{S\setminus\{s'\}}, T) = F(R_B, R_S, T) = z$. Then, when the true preference relation of s' is $R''_{s'}$, he pretends as if his preference relation is $R'_{s'}$, a contradiction to *strategy proofness*. Thus, $\Omega(R_B, R_{S\setminus\{s'\}}, R'_{s'}, T) = \sum_B p(R_b)$. The other cases can be proved similarly. ■

Proof. (Theorem 5) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *strategy proofness*, by the lemmas 3 and 7, f also satisfies those properties. Then, by Lemma 10, $f = U$.

The proofs of (5.2) and (5.3) are the same as the proof of (3.2) and (3.3), respectively. The only difference is that we use Lemma 8 instead of *peak-only*. The proof of (5.1) is the same as the proof of the property (4.3) of Theorem 4.

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